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AHARONOV INVARIANTS AND UNIVALENT FUNCTIONS

BY

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ABSTRACT

Several properties of a certain series of differential operators which are invariant under the Möbius group (Aharonov invariants) are proved, and in terms of this series new conditions for univalence and quasiconformal extendability of meromorphic functions are established.

I. Introduction

Let $\mathcal{M}(D)$ denote the linear space of meromorphic functions in a simply-connected domain D in $\hat{\mathbf{C}} = \mathbf{C} \cup \{\infty\}$. For every $f \in \mathcal{M}(D)$ we define Aharonov invariants $\{\psi_n(f,z)\}_{n=1}^{\infty}$ by means of the generating function

$$(1.1) \qquad \frac{f'(z)}{f(z)-f(\zeta)} - \frac{1}{z-\zeta} = \sum_{n=1}^{\infty} \psi_n(f,z)(\zeta - z)^{n-1}, \quad z \in D, \ |\zeta - z| < d(z, \partial D)$$

(cf. Aharonov [1]), where $d(z,\partial D)$ is the Euclidean distance of z to the boundary ∂D of D. Aharonov proved in [1] that this sequence has the following remarkable properties:

(i) $\psi_2(f,z) = \frac{1}{6}\{f,z\} = \frac{1}{6}[(f''/f')' - \frac{1}{2}(f''/f')^2](z)$, $z \in D$ and

$$(1.2) \qquad (n+1)\psi_n(f,z) = \psi'_{n-1}(f,z) + \sum_{j=2}^{n-2} \psi_j(f,z)\psi_{n-j}(f,z), \quad n = 3,4,\cdots.$$

(ii) For $n \geqq 2$ each $\psi_n(f,z)$ is invariant in the sense that

$$(1.3) \qquad \psi_n(g \circ f, z) = \psi_n(f,z) \quad \text{for every Möbius transformation } g.$$

(iii) THEOREM A. *Let $f \in \mathcal{M}(U)$ where $U = \{|\zeta| < 1\}$. Then f is univalent in U iff*

$$(1.4) \qquad \sup_{|\zeta|<1} \left\{ \sum_{n=2}^{\infty} (n-1) \left| \sum_{k=2}^{n} \binom{n-2}{k-2} (-\bar{\zeta})^{n-k} (1-|\zeta|^2)^k \psi_k(f,\zeta) \right|^2 \right\} = M^2(f) < \infty$$

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and also iff

$$\sup_{|\zeta|<1}\left\{\sup_{n\geqq 2}\left|\sum_{k=2}^{n}\binom{n-2}{k-2}(-\bar{\zeta})^{n-k}(1-|\zeta|^2)^k\psi_k(f,\zeta)\right|^2\right\}=M^\infty(f)<\infty \tag{1.5}$$

where both $M^2(f)$ *and* $M^\infty(f)$ *are bounded by* 1 *for univalent* f.

Here we generalize Theorem A and also refine its "only if" part for univalent functions with quasiconformal extensions.

II. Transformation formula

The proof of Theorem A is based on the following area inequality ([1] page 603)

$$\sum_{n=2}^{\infty}(n-1)|\psi_n(f,0)|^2\leqq 1, \tag{2.1}$$

which holds for every univalent $f\in\mathcal{M}(U)$, and on the transformation formula ([1], theorem 2)

$$\psi_n(f\circ g,z)=\sum_{k=2}^{n}\binom{n-2}{k-2}\frac{(-\bar{\zeta})^{n-k}(1-|\zeta|^2)^k}{(1+\bar{\zeta}z)^{n+k}}\psi_k(f,g(z)), \tag{2.2}$$

$$g(z)=\frac{z+\zeta}{1+\bar{\zeta}z},\quad |\zeta|<1.$$

In this section we generalize (2.2), and deduce simple univalence criteria from it.

Let g be analytic in D. Denote

$$\zeta=g(z),\quad \omega=g(z+w)-g(z)=\sum_{k=1}^{\infty}\frac{g^{(k)}(z)}{k!}w^k,\qquad z\in D,\quad |w|<d(z,\partial D)$$

and

$$\omega^l=(g(z+w)-g(z))^l=\sum_{k=l}^{\infty}A_{k,l}(g,z)w^k. \tag{2.3}$$

LEMMA 1. *Let* $g\in\mathcal{M}(D)$ *and* $f\in\mathcal{M}(g(D))$. *Then*

$$\psi_n(f\circ g,z)=\sum_{l=2}^{n}B_{n,l}(g,z)\psi_l(f,g(z))+\psi_n(g,z),\qquad z\in D,\quad n\geqq 2 \tag{2.4}$$

where

$$B_{n,l}(g,z)=g'(z)A_{n-1,l-1}(g,z). \tag{2.5}$$

PROOF. Setting $f \circ g$ instead of f in (1.1) we deduce

$$\frac{w(f\circ g)'(z)}{(f\circ g)(z+w)-(f\circ g)(z)} = 1-\sum_{n=1}^{\infty}\psi_n(f\circ g, z)w^n$$

$$= \frac{wg'(z)f'(\zeta)}{f(\zeta+\omega)-f(\zeta)} = \frac{wg'(z)}{\omega}\left(1-\sum_{l=1}^{\infty}\psi_l(f,\zeta)\omega^l\right)$$

$$= \frac{wg'(z)}{g(z+w)-g(z)} - g'(z)\sum_{l=1}^{\infty}\psi_l(f,g(z))\sum_{k=l-1}^{\infty}A_{k,l-1}(g,z)w^{k+1}$$

$$= 1-\sum_{n=1}^{\infty}\psi_n(g,z)w^n - g'(z)\sum_{k=0}^{\infty}w^{k+1}\sum_{l=1}^{k+1}A_{k,l-1}(g,z)\psi_l(f,g(z))$$

$$= 1-\sum_{n=1}^{\infty}\left\{\sum_{l=1}^{n}g'(z)A_{n-1,l-1}(g,z)\psi_l(f,g(z))+\psi_n(g,z)\right\}w^n$$

and since $A_{n-1,0}(g,z)=0$ for $n \geqq 2$, (2.4) and (2.5) follow. q.e.d.

REMARK. The coefficients $A_{k,l}(g,z)$ defined above are closely related to the so-called Bell polynomials appearing in Combinatorics (cf. [3], chapter 3.4). Jabotinsky [6] and Todorov [9] studied those coefficients in connection with Grunsky coefficients, and in [5] explicit formulas are obtained for Aharonov invariants and Bernoulli numbers in terms of Bell polynomials.

LEMMA 2. *If $g(z)=(az+b)/(cz+d)$, $ad-bc=1$, then*

$$\psi_n(f\circ g,z)=\sum_{l=2}^{n}\binom{n-2}{l-2}(-c)^{n-l}g'(z)^{(n+l)/2}\psi_l(f,g(z)),\quad n\geqq 2. \tag{2.4'}$$

PROOF. Every Möbius transformation g satisfies the identity

$$g(z)-g(\zeta)=(z-\zeta)g'(z)^{\frac{1}{2}}g'(\zeta)^{\frac{1}{2}}=\frac{z-\zeta}{(cz+d)(c\zeta+d)} \tag{2.6}$$

and hence

$$(g(z+w)-g(z))^l = g'(z)^{l/2}\sum_{k=0}^{\infty}\frac{[g'(z)^{l/2}]^{(k)}}{k!}w^{k+l}$$

$$=\sum_{n=l}^{\infty}\frac{g'(z)^{l/2}}{(n-l)!}[g'(z)^{l/2}]^{(n-l)}w^n.$$

But $g'(z)=(cz+d)^{-2}$ and therefore

$$[g'(z)^{l/2}]^{(n-l)}=[(cz+d)^{-l}]^{(n-l)}=\frac{(n-1)!}{(l-1)!}\frac{(-c)^{n-l}}{(cz+d)^n}.$$

Hence

$$A_{n,l}(g,z)=\binom{n-1}{l-1}(-c)^{n-l}g'(z)^{(n+l)/2}. \tag{2.7}$$

As g is Möbius $\psi_n(g,z)=0$. Thus (2.4), (2.5) and (2.7) imply (2.4′). q.e.d.

THEOREM 1. *Let $f\in\mathcal{M}(U)$. If f is univalent then*

$$\begin{aligned}(1-|z|^2)^n|\psi_n(f,z)|&\leqq p_{n-2}(|z|)\\&=\sum_{k=2}^{n}\binom{n-2}{k-2}\frac{|z|^{n-k}}{\sqrt{k-1}},\qquad |z|<1,\quad n\geqq 2.\end{aligned} \tag{2.8}$$

PROOF. Let $g(\zeta)=(\zeta+z)/(1+\bar z\zeta)$. Then $g(0)=z$ and by (2.4′)

$$\begin{aligned}\psi_n(f,z)&=\psi_n(f\circ g\circ g^{-1},g(0))\\&=\sum_{k=2}^{n}\binom{n-2}{k-2}\left(\frac{\bar z}{\sqrt{1-|z|^2}}\right)^{n-k}(1-|z|^2)^{-(n+k)/2}\psi_k(f\circ g,0)\\&=(1-|z|^2)^{-n}\sum_{k=2}^{n}\binom{n-2}{k-2}\bar z^{n-k}\psi_k(f\circ g,0).\end{aligned} \tag{2.9}$$

But since f is univalent in U, $f\circ g$ is also univalent there, and from (2.1) we deduce that

$$|\psi_k(f\circ g,0)|\leqq\frac{1}{\sqrt{k-1}},\qquad k\geqq 2. \tag{2.10}$$

Thus (2.9) and (2.10) yield (2.8). q.e.d.

In [4] the following improvement on (2.1) was deduced:

$$\sum_{n=2}^{\infty}(n-1)|\psi_n(f,0)|^2\leqq\|\mu\|_\infty^2\quad\text{(cf. [7])} \tag{2.1′}$$

for every f which is univalent in U and has a μ-quasiconformal extension into $\hat{\mathbf{C}}\setminus U$. This implies at once:

THEOREM 1′. *If f is univalent in U and has a μ-quasiconformal extension into $\hat{\mathbf{C}}\setminus U=\{|\zeta|\geqq 1\}$, then*

$$(1-|z|^2)^n|\psi_n(f,z)|\leqq p_{n-2}(|z|)\|\mu\|_\infty,\qquad |z|<1,\quad n\geqq 2. \tag{2.8′}$$

From the Definition (2.3) of $A_{k,l}(g,z)$ one can easily derive (cf. Jabotinsky [6]):

LEMMA 3. *Let $g \in \mathcal{M}(D)$ and $f \in \mathcal{M}(g(D))$. Then*

$$A_{k,m}(f \circ g, z) = \sum_{l=m}^{k} A_{l,m}(f, g(z)) A_{k,l}(g, z). \tag{2.11}$$

In particular, for $m = 1$ we obtain the Faá-di-Bruno formula

$$\frac{1}{k!}(f \circ g)^{(k)}(z) = \sum_{l=0}^{k} A_{k,l}(g, z) \frac{f^{(l)}(g(z))}{l!} \quad \text{(cf. [3]).} \tag{2.11$'$}$$

III. Aharonov sequence and univalence

In view of Theorem A we consider, in this section, the sequence $\{\psi_n(f, z)\}_{n=2}^{\infty}$, for every $f \in \mathcal{M}(D)$, as one entity, namely as a (row-) vector $\Psi_f(z)$ of meromorphic functions. Then one can write the transformation formula (2.4) in the following matrix form:

$$\Psi_{f \circ g}(z) = \Psi_f(g(z)) Q_g(z) + \Psi_g(z) \tag{2.4*}$$

where $Q_g(z)$ is the semi-infinite upper-triangular matrix

$$Q_g(z) = (B_{n,l}(g, z))_{n,l=2}^{\infty} \tag{2.5$'$}$$

with the $B_{n,l}(g, z)$ as defined in (2.5) for $n \geqq l \geqq 2$ and $B_{n,l}(g, z) = 0$ for $n < l$. Thus every conformal mapping g in D defines the following operator:

$$Q_g^* : \mathcal{B}(g(D)) \ni \Phi \mapsto (\Phi \circ g) Q_g^* \in \mathcal{B}(D)$$

where $\mathcal{B}(D)$ is the vector space of all the sequences (row-vectors) $\Phi = (\phi_n)_{n=2}^{\infty}$ of meromorphic functions in D.

Now let D be a hyperbolic simply-connected domain, with the Poincaré-metric $\rho_D(z)$ (normalized to have a constant curvature -4). Again in view of Theorem A we denote by $\mathcal{B}^2(D)$ and $\mathcal{B}^{\infty}(D)$ the normed subspaces of all the sequences $\Phi \in \mathcal{B}(D)$ which are bounded either in the norm

$$\|\Phi\|_{2,D} = \sup_{g: U \to D} \left(\sum_{n=2}^{\infty} (n-1) |(Q_g^* \Phi)_n(0)|^2 \right)^{\frac{1}{2}} \quad \text{in } \mathcal{B}^2(D) \tag{3.1}$$

or in the norm

$$\|\Phi\|_{\infty,D} = \sup_{g: U \to D} \left(\sup_{n \geqq 2} |(Q_g^* \Phi)_n(0)| \right) \quad \text{in } \mathcal{B}^{\infty}(D) \tag{3.2}$$

where in both norms the suprema are taken over all the conformal mappings g of the unit disc U onto D.

The following lemma sums up some obvious facts about $\mathscr{B}^2(D)$ and $\mathscr{B}^\infty(D)$.

LEMMA 4. (i) *If* $\Phi \in \mathscr{B}^2(D)$ *then*

$$\|\Phi\|_\infty \leqq \|\Phi\|_2 \tag{3.3}$$

and therefore $\mathscr{B}^2(D) \subseteq \mathscr{B}^\infty(D)$ *continuously.*

(ii) *If* $\Phi = (\phi_n)_{n=2}^\infty \in \mathscr{B}^\infty(D)$, *then*

$$\|\Phi\|_\infty \geqq \sup_{g:U\to D} |(Q_g^*\Phi)_2(0)| = \sup_{\zeta\in D} \rho_D(\zeta)^{-2}|\phi_2(\zeta)| \equiv \|\phi_2\|_{2,D} \tag{3.4}$$

and therefore the "projection"

$$\mathscr{B}^p(D) \ni (\phi_n)_{n=2}^\infty \mapsto \phi_2 \in \mathscr{B}_2(D) = \{\phi \text{ analytic in } D, \|\phi\|_{2,D} < \infty\}$$

is continuous.

(iii) *If* $h : D \to \tilde{D}$ *is a conformal mapping, then* Q_h^* *is an isometric-isomorphism of* $\mathscr{B}^p(\tilde{D})$ *onto* $\mathscr{B}^p(D)$, $p = 2, \infty$.

PROOF. Part (i) is self-evident. (ii) follows from the identity $\rho_D(\zeta) = |g'(0)|^{-1}$ for every conformal mapping $g : U \to D$ with $\zeta = g(0)$. (iii) follows from Jabotinsky multiplication formula (2.11) which has the matrix form

$$Q_{f\circ g}(z) = Q_f(g(z))Q_g(z) \tag{2.11$^+$}$$

and also the operator form

$$Q_{f\circ g}^* = Q_g^* \circ Q_f^*. \tag{2.11*}$$

Thus, since $Q_{\mathrm{id}} = I =$ the unit matrix, $Q_{h^{-1}}^* = (Q_h^*)^{-1}$ for univalent h in D, and therefore Q_h^* is an isomorphism of $\mathscr{B}(\tilde{D})$ onto $\mathscr{B}(D)$.

Finally, if $\Phi \in \mathscr{B}^\infty(\tilde{D})$, then

$$\|Q_h^*\Phi\|_{\infty,D} = \sup_{g:U\to D}\left\{\sup_{n\geqq 2}|(Q_g^*\circ Q_h^*\Phi)_n(0)|\right\}$$

$$= \sup_{h\circ g:U\to D}\left\{\sup_{n\geqq 2}|(Q_{h\circ g}^*\Phi)_n(0)|\right\} = \|\Phi\|_{\infty,\tilde{D}}.$$

Similarly we prove that $\|Q_h^*\Phi\|_{2,D} = \|\Phi\|_{2,\tilde{D}}$. q.e.d.

Notice that by the recursion formula (1.2), Ψ_f belongs to the set

$$\mathscr{B}_A(D) = \left\{\Phi = (\phi_n)_{n=2}^\infty \in \mathscr{B}(D) : (n+1)\phi_n = \phi_{n-1}' + \sum_{j=2}^{n-2}\phi_j\phi_{n-j},\ n = 3, 4, \cdots\right\}.$$

THEOREM 2. *Let D be a hyperbolic simply-connected domain.*

(i) *If f is meromorphic univalent in D, then*

$$\Psi_f \in \mathcal{B}^2_A(D) = \mathcal{B}^2(D) \cap \mathcal{B}_A(D)$$

and

$$\|\Psi_f\|_{2,D} \leqq 2. \tag{3.5}$$

(ii) *If $\Phi \in \mathcal{B}^\infty_A(D) = \mathcal{B}^\infty(D) \cap \mathcal{B}_A(D)$, then $\Phi = \Psi_f$ for some meromorphic univalent f in D.*

(iii) *A meromorphic function f is univalent in D iff $\Psi_f \in \mathcal{B}^p_A(D)$, $p = 2, \infty$.*

Proof. If D is a disc, then $\Psi_g = 0$ for any conformal mapping g of U onto D, and $f \circ g$ is univalent in U whenever f is univalent in D. Then by (2.1) and (2.4*)

$$\sum_{n=2}^{\infty} (n-1)|\psi_n(f \circ g, 0)|^2 = \sum_{n=2}^{\infty} (n-1)|(Q^*_g\Psi_f)_n(0)|^2 \leqq 1, \tag{1.4'}$$

i.e.

$$\|\Psi_f\|_{2,D} \leqq 1. \tag{3.5'}$$

Now, if f is univalent in a hyperbolic simply-connected domain D, and g is any conformal mapping of U onto D, then by Lemma 4 (iii), formula (2.4*) and (3.5')

$$\|\Psi_f\|_{2,D} = \|Q^*_g\Psi_f\|_{2,U} \leqq \|\Psi_{f\circ g}\|_{2,U} + \|\Psi_g\|_{2,U} \leqq 2.$$

(ii) Conversely, if $\Phi = (\phi_n)_{n=2}^{\infty} \in \mathcal{B}^\infty_A(D)$, then by (3.4), $\phi_2 \in \mathcal{B}_2(D)$. Hence, by a well-known property of the Schwarzian derivative $\phi_2(z) = \psi_2(f, z) = \frac{1}{6}\{f, z\}$ for some locally univalent meromorphic function f in D. Therefore $\Phi = (\phi_n)_{n=2}^{\infty} = (\psi_n(f))_{n=2}^{\infty} = \Psi_f$ by the definition of $\mathcal{B}_A(D)$.

Now, if $\Phi \in \mathcal{B}^\infty_A(U)$, then for every Möbius self-mapping g of U we have

$$|(Q^*_g\Phi)_n(0)| = |(Q^*_g\Psi_f)_n(0)| = |\psi_n(f \circ g, 0)| \leqq \|\Phi\|_{\infty,U} < \infty, \quad n \geqq 2.$$

But this is exactly Aharonov's sufficient condition (1.5) for univalence of f in U.

Finally, if $\Phi \in \mathcal{B}^\infty_A(D)$, so that $\Phi = \Psi_f$ for some $f \in \mathcal{M}(D)$, then by (2.4*), (3.3), Lemma 4 (iii) and (3.5')

$$\|\Psi_{f\circ g}\|_{\infty,U} \leqq \|Q^*_g\Psi_f\|_{\infty,U} + \|\Psi_g\|_{2,U} \leqq \|\Phi\|_{\infty,D} + 1 < \infty$$

for any conformal mapping of U onto D. Hence $\Psi_{f\circ g} \in \mathcal{B}^\infty_A(U)$ so that $f \circ g$ is univalent in U, and therefore f is univalent in $D = g(U)$.

Part (iii) follows at once from parts (i) and (ii), and Lemma 4 (i). q.e.d.

COROLLARY 1. $\mathcal{B}^2_A(D) = \mathcal{B}^\infty_A(D) \subset \{\Phi \in \mathcal{B}^2(D) : \|\Phi\|_{2,D} \leqq 2\}$.

PROOF. $\mathcal{B}^2_A(D) \subseteq \mathcal{B}^\infty_A(D)$ by Lemma 4 (i). Conversely if $\Phi \in \mathcal{B}^\infty_A(D)$, then $\Phi = \Psi_f$ for a univalent f, and hence $\|\Phi\|_{2,D} \leqq 2$. q.e.d.

IV. Invariant sequence and quasiconformal extension

Combining (2.1′) with (2.2) one can prove the following quasiconformal refinement of the "only if" part of Theorem A:

LEMMA 5. *If f is univalent in the unit disc U and has a μ-quasiconformal extension $\hat{\mathbf{C}} \setminus U$, then*

$$(4.1)\quad \|\Psi_f\|_{2,U} = \sup_{|\zeta|<1} \left\{ \sum_{n=2}^{\infty} (n-1) \left| \sum_{k=2}^{n} \binom{n-2}{k-2} (-\bar{\zeta})^{n-k} (1-|\zeta|^2)^k \psi_k(f,\zeta) \right|^2 \right\}^{\frac{1}{2}} \leqq \|\mu\|_\infty$$

and therefore also

$$(4.1') \qquad \|\Psi_f\|_{\infty,U} \leqq \|\mu\|_\infty.$$

In this section we generalize (4.1′).

THEOREM 3. *Let D_1 and D_2 be complementary simply-connected domains in $\hat{\mathbf{C}}$, and let f_0 and f_1 be two conformal mappings in D_2 with μ_0 and μ_1-quasiconformal extensions, respectively, into D_1, and denote $\Psi_i = \Psi_{f_i}$, $i = 0, 1$. Then*

$$(4.2) \qquad \|\Psi_1 - \Psi_0\|_{\infty, D_2} \leqq 2 \left\| \frac{\mu_1 - \mu_0}{1 - \bar{\mu}_0 \mu_1} \right\|_\infty.$$

PROOF. Assume first that $D_2 = U$ is the unit disc. "Join" μ_0 with μ_1 by the "analytic disc":

$$\mu(\zeta) = \frac{\zeta\mu + \mu_0}{1 + \zeta\bar{\mu}_0\mu}, \quad |\zeta| < \|\mu\|_\infty^{-1} \qquad \text{where } \mu = \frac{\mu_1 - \mu_0}{1 - \bar{\mu}_0\mu_1}$$

such that

$$(4.3) \qquad \mu(0) = \mu_0, \quad \mu(1) = \mu_1 \quad \text{and} \quad \|\mu\|_\infty = \|\mu_{f_1 \circ f_0^{-1}}\|_\infty.$$

For every Beltrami coefficient μ supported in D_1, let f_μ be the unique μ-quasiconformal automorphism of $\hat{\mathbf{C}}$ which fixes $\{0, 1, \infty\}$, and denote $\psi_n^\mu(z) = \psi_n(f_\mu, z)$, $z \in D_2$. Denote $\psi_n^{\mu(\zeta)} = \psi_n(\zeta; z)$ and $\Psi^\zeta(z) = (\psi_n(\zeta; z))_{n=2}^\infty$. From the uniqueness theorem for Beltrami equation and (4.3) we deduce that $f_{\mu(0)}(z)$ and

$f_{\mu(1)}(z)$ differ from $f_0(z)$ and $f_1(z)$, respectively, by left composition with Möbius transformations. Therefore, the invariance property (1.3) yields

$$\Psi^0(z)=\Psi_0(z) \quad \text{and} \quad \Psi^1(z)=\Psi_1(z).$$

Since $\mu(\zeta)$ is an analytic function of ζ in the disc $D=\{|\zeta|<\|\mu\|_\infty^{-1}\}$, the generating function

$$\frac{f_z'(\zeta;z)}{f(\zeta;z)-f(\zeta;w)}-\frac{1}{z-w}=\sum_{n=1}^{\infty}\psi_n(\zeta;z)(w-z)^{n-1} \quad (\text{where } f(\zeta;z)=f_{\mu(\zeta)}(z))$$

is an analytic function of $\zeta\in D$ (see Ahlfors–Bers [2]), and therefore all the invariants $\psi_n(\zeta;z)$ are analytic functions of $\zeta\in D$ for every $z\in U$.

For every Möbius self-mapping g of U denote

$$\begin{aligned} a_n(g;\zeta)&=(Q_g^*(\Psi^\zeta-\Psi^0))_n(0)\\ &=\sum_{l=2}^{n}B_{n,l}(g,0)[\psi_l(\zeta;g(0))-\psi_l(0;g(0))], \qquad n\geqq 2. \end{aligned}$$

Then $a_n(g;\zeta)$ is analytic in D, $a_n(g;0)=0$ and by Lemma 5 we have for every $n\geqq 2$

$$\begin{aligned} |a_n(g;\zeta)|&\leqq\left(\sum_{k=2}^{\infty}(k-1)|a_k(g;\zeta)|^2\right)^{\frac12}\\ &\leqq\|\Psi^\zeta-\Psi_0\|_{2,U}\leqq 1+\|\mu_0\|_\infty, \quad \zeta\in D. \end{aligned}$$

Applying Schwarz' lemma we conclude that

$$|a_n(g;\zeta)|\leqq(1+\|\mu_0\|)\|\mu\|\,|\zeta|, \qquad |\zeta|<\|\mu\|^{-1}, \quad n\geqq 2$$

and for $\zeta=1<\|\mu\|^{-1}$ we deduce

$$|a_n(g;1)|=|(Q_g^*(\Psi_1-\Psi_0))_n(0)|\leqq(1+\|\mu_0\|)\|\mu\|, \qquad n\geqq 2$$

for every conformal self-mapping g of U. Therefore

$$(4.2') \quad \|\Psi_1-\Psi_0\|_{\infty,U}=\sup_{g:U\to U}\left(\sup_{n\geqq 2}|a_n(g;\zeta)|\right)\leqq(1+\|\mu_0\|)\|\mu\|<2\|\mu_{f_1\circ f_0^{-1}}\|.$$

Now assume $C=\partial D_1=\partial D_2$ is a quasicircle, so that any conformal mapping g of U onto D_2 has a quasiconformal extension into $\hat{\mathbf{C}}\setminus U$, also denoted by g. Then

$$\mu_{f_1\circ f_0^{-1}}=\mu_{f_1\circ g\circ(f_0\circ g)^{-1}} \quad \text{in } f_0(D_1)=f_0\circ g(\hat{\mathbf{C}}\setminus U)$$

and in U we have

$$Q_g^*(\Psi_1 - \Psi_0) = (\Psi_{f_1 \circ g} - \Psi_g) - (\Psi_{f_0 \circ g} - \Psi_g) = \Psi_{f_1 \circ g} - \Psi_{f_0 \circ g}.$$

Hence, the isometry property of Q_g^* and (4.2′) imply

$$\|\Psi_1 - \Psi_0\|_{\infty, D_2} = \| Q_g^*(\Psi_1 - \Psi_0)\|_{\infty, U} = \|\Psi_{f_1 \circ g} - \Psi_{f_0 \circ g}\|_{\infty, U} \leqq 2\|\mu_{f_1 \circ f_0^{-1}}\|_\infty. \tag{4.2″}$$

Finally, if the common boundary $C = \partial D_1 = \partial D_2$ is not a quasicircle, we denote for every conformal mapping $g : U \to D_2$ and $0 < r < 1$

$$g_r(z) = g(rz), \quad D_2(r) = g_r(U) \subset D_2, \quad C_r = \partial D_2(r) \quad \text{and} \quad D_1(r) = \hat{\mathbf{C}} \setminus \overline{D_2(r)}.$$

Then C_r is a quasicircle, and since $(Q_{g_r}^*\Phi)_n(0) = r^n (Q_g^*\Phi)_n(0)$ we deduce from (4.2″)

$$r^n |(Q_g^*(\Psi_1 - \Psi_0))_n(0)| = |(Q_{g_r}^*(\Psi_1 - \Psi_0))_n(0)| \leqq \|\Psi_1 - \Psi_0\|_{\infty, D_2(r)} < 2\|\mu_{f_1 \circ f_0^{-1}}\|_\infty.$$

This completes the proof by passing to a limit $r \to 1$. q.e.d.

REMARK. Using the same technique, O. Lehto has proved in [8] an analogous result for the Schwarzian derivative, which is now included in Theorem 3.

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